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Schur Indices in Finite Groups*

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1. INTRODUCTION

Suppose G is a finite group with exponent n and F is a field of characteristic zero. We write $F[\sqrt[n]{1}]$ to denote the splitting field of $X^n - 1$ over F. A well-known theorem of R. Brauer (16.3 of [6]) asserts that $F[\sqrt[n]{1}]$ is a splitting field for G. Consequently, if one is interested in the irreducible F-representations of G, it is natural to study the field extension, $F[\sqrt[n]{1}]/F$. We prove the following result.

THEOREM 1. *Let G be a finite group with exponent n and let F be a field of characteristic zero. Suppose for some prime p, that a Sylow p-subgroup P of the Galois group* $\mathrm{Gal}(F[\sqrt[n]{1}]/F)$ *is cyclic. If $2 \mid |P|$, assume also that $\sqrt{-1} \in F$. Then the Schur index over F of every irreducible character of G is relatively prime to p.*

This theorem includes results of L. Solomon [9] and P. Fong [7] as special cases. (See 16.4 and 16.5 of [6].) The generalization of Fong's result to the exponent rather than the order of G was also obtained recently by T. Yamada [10].

Theorem 1 is obtained using a reduction theorem similar in spirit to a result of Brauer [3] (See 16.1 of [6]). To state this result, we first make the following

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DEFINITION. Let F be a field of characteristic zero. An *F-triple* is an ordered triple (H, X, θ), where

(a) H is a finite group with a cyclic, normal, self-centralizing subgroup X,

(b) θ is a faithful (absolutely) irreducible character of H,

(c) θ_X is afforded by an irreducible $F(\theta)$-representation.

Following [6] we let $m_F(\chi)$ denote the Schur index over F of the (absolutely) irreducible character χ, and we write $\mathrm{Irr}(G)$ for the set of (absolutely) irreducible characters of G.

THEOREM 2. *Let G be a finite group and F a field of characteristic zero. Suppose $\chi \in \mathrm{Irr}(G)$ and $p^a \mid m_F(\chi)$ for some prime p. Then there exists an F-triple (H, X, θ) such that*

(a) *H is a section of G,*

(b) *H/X is a p-group,*

(c) $p^a \mid m_F(\theta)$,

(d) $p \nmid |F(\chi, \theta) : F(\chi)|$.

We say that a group is *quasielementary* if it has a cyclic normal p-complement for some prime p. (These groups have also been called "hyperelementary.") The key step in the proof of Theorem 2 is to show that a minimal counterexample is quasielementary. Although this is immediate from standard theorems, we take this opportunity to present a more elementary proof. We use the following:

THEOREM 3. (Solomon). *Let $\mathscr{H}$ be the set of quasielementary subgroups of the finite group G. Then there exist rational integers $\{a_H \mid H \in \mathscr{H}\}$ such that*

$$1_G = \sum_{H \in \mathscr{H}} a_H (1_H)^G.$$

We give a surprisingly easy proof of Solomon's (unpublished) result. We note that Brauer's Theorem on induced characters [4] is a reasonably direct consequence of Theorem 3, and we sketch a proof.

2. F-TRIPLES

Let (H, X, θ) be an F-triple with $|X| = r$ and let λ be an irreducible constituent of θ_X. Since θ is faithful and X is cyclic, it follows that λ is a faithful linear character and in particular, $F(\lambda) = F[\sqrt[r]{1}]$. Since Schur indices of linear characters are trivial, the hypothesis that θ_X is afforded by an

$F(\theta)$-irreducible representation means that $\theta_X = \Sigma_\sigma \lambda^\sigma$, where the sum ranges over all $\sigma \in \mathfrak{G} = \mathrm{Gal}(F(\lambda)/F(\theta))$. We call the extension $F(\lambda)/F(\theta)$ the *associated field extension* of (H, X, θ). For $h \in H$, there exists a unique $\sigma_h \in \mathfrak{G}$ such that

$$\lambda^{h^{-1}} = \lambda^{\sigma_h}.$$

It is easy to check that the map $h \mapsto \sigma_h$ is an epimorphism from H to $\mathfrak{G}$ whose kernel is $I(\lambda)$, the inertia group of λ. However, since λ is faithful, the hypothesis that X is self-centralizing implies that $I(\lambda) = X$. This yields $H/X \simeq \mathfrak{G}$ and also shows that $\theta = \lambda^G$. We summarize these observations as follows:

LEMMA 2.1. *Let (H, X, θ) be an F-triple with $|X| = r$ and let λ be an irreducible constituent of θ_X. Then*

(a) $F(\lambda) = F[\sqrt[r]{1}]$,
(b) $\theta = \lambda^G$,
(c) $H/X \simeq \mathrm{Gal}(F(\lambda)/F(\theta))$.

We turn now to a discussion of the Schur index of θ over F in the context of the theory of central-simple algebras. (Readers unfamiliar with this material may skip this discussion and proceed immediately to the statement of the main result of this section, Theorem 2.2.) Given the F-triple (H, X, θ), let $F_1 = F(\theta)$ and let A be the ideal of $F_1[H]$ generated by the central idempotent

$$e = [\theta(1)/|H|] \sum_{h \in H} \theta(h^{-1})h.$$

Then A is a central-simple F_1-algebra and $m_F(\theta)$ is just the index of A. The interesting point about F-triples is that we have an explicit crossed product presentation for A (see [8], p. 108). Namely, let $E = eF_1[X]$, then E is a field because θ_X is afforded by an F_1-irreducible representation. Clearly, $E \simeq F_1[\sqrt[r]{1}] \simeq F(\lambda)$. The elements $\{eh \mid h \in H\}$ generate A (as an F_1-vector space), normalize E, and by virtue of (2.1), induce the action of $\mathrm{Gal}(E/F_1)$ on E. A dimension count yields:

$$\dim_{F_1}(A) = \theta(1)^2 = (\lambda^G(1))^2 = |H/X|^2 = |E:F_1|^2.$$

Thus A is explicitly presented as a crossed product. The mapping $H \to eH$ embeds X into the multiplicative group, E^*, of E and identifies H/X with $\mathrm{Gal}(E/F_1)$. These identifications induce a map

$$\varphi: H^2(H/X, X) \to H^2(\mathrm{Gal}(E/F_1), E^*).$$

If $\alpha \in H^2(H/X, X)$ is the class which defined the group H, then $\varphi(\alpha)$ is the class which defines the crossed product A.

Remark. It is obvious that the order of $\varphi(\alpha)$ (in the group $H^2(\mathrm{Gal}(E/F_1), E^*)$) divides the order of X. If F is a number field, it follows from deep results on division algebras that the index of A equals the order of $\varphi(\alpha)$ (see [1, p. 149]). In particular, $m_F(\theta) \mid |X|$.

We now consider the special case where H/X is cyclic. Let $Y = \langle y \rangle$ be a cyclic supplement to X in H, let $Y_0 = Y \cap X$, and let $m = |H/X|$. Then $(ey)^m$ is a primitive $|Y_0|$th root of unity in E, and the standard necessary and sufficient condition for the cyclic crossed product $A = \langle E, ey \rangle$ to split is that $(ey)^m$ lie in the image of the norm map $N_{E/F_1} : E^* \to F_1^*$ (see [1, p. 75]). This proves the following.

THEOREM 2.2. *Suppose (H, X, θ) is an F-triple with H/X cyclic. Let Y be a cyclic supplement to X in H and put $n = |Y \cap X|$. Let E/F_1 be the associated field extension. Then $m_F(\theta) = 1$ iff the image of the norm map $N_{E/F_1} : E^* \to F_1^*$ contains a primitive nth root of unity.*

We now outline a direct proof of the sufficiency of the above-stated norm condition which is independent of the theory of algebras. The necessity can also be proved directly, but it is not required in this paper. Suppose, then, that λ is an irreducible constituent of θ_X, and set $E = F(\lambda)$ and $F_1 = F(\theta)$. Let $Y = \langle y \rangle$, $X = \langle x \rangle$, and $m = |H/X|$. Then a presentation for H is given by

$$H = \langle x, y \mid x^r = 1, y^m = x^s, y^{-1}xy = x^t \rangle$$

where s and t are appropriate integers. In the notation of (2.2), $y^m = x^s$ is a central element of order n. Since θ is a faithful irreducible character, F_1 contains a primitive nth root of unity, ϵ. Without loss, we may take $\epsilon = \lambda(x)^s$. From (2.1) it follows that $\mathrm{Gal}(E/F_1)$ is generated by an element σ such that $\lambda(x)^\sigma = \lambda^{y^{-1}}(x) = \lambda(x^y) = \lambda(x^t) = \lambda(x)^t$. We are assuming that there exists $\alpha \in E$ with $\prod_{i=0}^{m-1} \alpha^{\sigma^i} = \epsilon = \lambda(x)^s$. Define F_1-linear transformations $\hat{x}$, $\hat{y}$ on E via

$$\beta\hat{x} = \beta\lambda(x),$$
$$\beta\hat{y} = \beta^\sigma\alpha$$

for all $\beta \in E$. Then direct calculation yields $\hat{x}^r = 1$, $\hat{y}^m = \hat{x}^s$, $\hat{y}^{-1}\hat{x}\hat{y} = \hat{x}^t$. It follows that there exists an F_1-representation $\mathfrak{X}$ with $\mathfrak{X}(x) = \hat{x}$, $\mathfrak{X}(y) = \hat{y}$. Let $\hat{\theta}$ be the character afforded by $\mathfrak{X}$. Then $\hat{\theta}(1) = |E : F_1| = |H/X|$ and λ is obviously a constituent of $\hat{\theta}_X$. We conclude that $\hat{\theta} = \lambda^G = \theta$ is afforded by an F_1-representation.

3. Quasielementary Subgroups

Recall that a group is called quasielementary if it has a cyclic normal p-complement for some prime, p. Let G be a finite group and let R be the set of $\mathbf{Z}$-linear combinations of permutation characters of the form $(1_H)^G$ where $H \subseteq G$ is quasielementary.

Lemma 3.1. *In the above situation, R is a ring of $\mathbf{Z}$-valued functions (with pointwise addition and multiplication).*

Proof. It is clear that R is closed under addition and consists of $\mathbf{Z}$-valued functions. Thus, it suffices to show that $(1_H)^G(1_K)^G \in R$ for quasielementary $H, K \subseteq G$. Write $\theta = (1_H)^G$. Then $\theta(1_K)^G = (\theta_K)^G$. Now θ_K is a permutation character of K and hence we may write $\theta_K = \sum_{i=1}^s (1_{K_i})^K$, where $\{K_i\}$ is some family of (not necessarily distinct) subgroups of K. Thus $(1_H)^G(1_K)^G = (\theta_K)^G = \sum_{i=1}^s (1_{K_i})^G$. Since every subgroup of K is quasielementary, the result follows.

Theorem 3 asserts that $1_G \in R$. To prove this, we use the following special case of a lemma of Banashewski [2].

Lemma 3.2. *Suppose R is a ring of $\mathbf{Z}$-valued functions on a finite set Ω. If $1_\Omega \notin R$, then there exists $\alpha \in \Omega$ and a prime, $p \in \mathbf{Z}$, such that $p \mid f(x)$ for every $f \in R$.*

Proof. For $\alpha \in \Omega$, let $I_\alpha = \{f(\alpha) \mid f \in R\}$. Then I_α is a subring of $\mathbf{Z}$. It suffices to assume that $I_\alpha = \mathbf{Z}$ for all $\alpha \in \Omega$ and to show that $1_\Omega \in R$.

Since $1 \in I_\alpha$ for each $\alpha \in \Omega$, we may choose $f_\alpha \in R$ with $f_\alpha(\alpha) = 1$. Then $\Pi_\alpha(1_\Omega - f_\alpha) = 0$. Expanding this yields 1_Ω as a linear combination of products of various f_α, so $1_\Omega \in R$.

Proof of Theorem 3. By Lemmas 3.1 and 3.2, it suffices to show that given $g \in G$ and a prime $p \in \mathbf{Z}$, there exists a quasielementary subgroup $H \subseteq G$ such that $p \nmid (1_H)^G(g)$. Let C be the p-complement in $\langle g \rangle$ and put $N = \mathbf{N}_G(C)$. Let H/C be a Sylow p-subgroup of N/C. Then H is quasielementary.

Now $(1_H)^G(g)$ is the number of cosets Hx in G with $Hxg = Hx$. However, if $Hxg = Hx$, then $C^{x^{-1}} \subseteq \langle g \rangle^{x^{-1}} \subseteq H$. Since C is the unique subgroup of H of its order, it follows that $x \in \mathbf{N}(C) = N$. Thus $(1_H)^G(g) = (1_H)^N(g)$. Since $C \lhd N$ and $C \subseteq H$, we see that C acts trivially on the cosets of H. Since $\langle g \rangle/C$ is a p-group, we therefore have $(1_H)^N(g) \equiv |N : H| \not\equiv 0 \pmod p$ and the proof is complete.

We digress at this point to indicate how to derive Brauer's Theorem on induced characters from Theorem 3. The key point of the proof (see [4]) is to write 1_G as a $\mathbf{Z}$-linear combination of characters induced from characters of elementary subgroups of G. From this it follows that every character of G has this form. By Theorem 3, we may assume that G is quasielementary and working by induction, it suffices to show that if G is not elementary, then 1_G is a $\mathbf{Z}$-linear combination of characters induced from proper subgroups. This may be done as follows.

Write $G = CP$ where $C \lhd G$ is a cyclic p'-group and P is a p-group. Assume $C_0 = \mathbf{C}_C(P) < C$ and let $H = PC_0 < G$. Suppose λ is a linear constituent of $((1_H)^G)_C = (1_{C_0})^C$. If λ is invariant in G then $[C, P] \subseteq \ker \lambda$. Since also $C_0 \subseteq \ker \lambda$ and $C_0[C, P] = C$, it follows that $\lambda = 1_C$. Therefore, we may write $(1_H)^G = 1_G + \Sigma\chi_i$ where the χ_i are irreducible characters of G with nonhomogeneous restrictions to C. It follows that the χ_i are induced from proper subgroups as desired.

4. Reduction Theorem

Proof of Theorem 2. Suppose $\chi \in \mathrm{Irr}(G)$ and $p^a \mid m_F(\chi)$. Proceeding by induction on $|G|$, we may assume that χ is faithful. We recall that condition (c) in the definition of an F-triple is equivalent to the assertion that all of the linear constituents of θ_X are conjugate by field automorphisms which fix F. In particular, any $F(\chi)$-triple is an F-triple and we may assume without loss that $F(\chi) = F$.

Suppose that for some $K \subseteq G$ we can find $\theta \in \mathrm{Irr}(K)$ such that $p \nmid |F(\theta) : F|$ and $p \nmid [\chi, \theta^G]$. Then $p^a \mid m_F(\theta)$. To see this, note that otherwise $p^{a-1}m\theta$ is afforded by an $F(\chi)$ representation for some m not divisible by p, and thus $p^{a-1}m[\chi, \theta^G] \mid F(\theta) : F \mid \chi$ is afforded by an F-representation of G by 11.4 and 11.1 of [6]. This contradicts the hypothesis that $p^a \mid m_F(\chi)$. By the inductive hypothesis, we may assume that no θ as above exists for any proper subgroup K.

Theorem 3 yields $1_G = \sum_H a_H(1_H)^G$, where H runs over the set $\mathscr{H}$ of quasi-elementary subgroups of G. Thus

$$\chi = \sum_{H \in \mathscr{H}} a_H(\chi_H)^G$$

and

$$1 = [\chi, \chi] = \sum_{H \in \mathscr{H}} a_H[\chi, (\chi_H)^G].$$

Choose $H \in \mathscr{H}$ such that $p \nmid [\chi, (\chi_H)^G]$ and write $\chi_H = \sum \Delta_i$, where the Δ_i are orbit sums under the action of $\mathfrak{G} = \mathrm{Gal}(F[\sqrt[n]{1}]/F)$. Now choose Δ_i so that $p \nmid [\chi, (\Delta_i)^G]$ and let θ be an irreducible constituent of Δ_i. Then since

$[\chi, \theta^G] = [\chi, (\theta^\sigma)^G]$ for all $\sigma \in \mathfrak{G}$, it follows that $p \nmid [\chi, \theta^G]$ and that the size of the $\mathfrak{G}$-orbit $\mathfrak{O}$ containing θ is prime to p. Since $|\mathfrak{O}| = |F(\theta) : F|$, we conclude that H is not a proper subgroup and hence G is quasielementary.

Now G has a cyclic normal q-complement, C, for some prime q and thus $\chi(1)$ is a power of q. Since $m_F(\chi) \mid \chi(1)$, we have $q = p$. Choose $X \supseteq C$, $X \lhd G$ maximal such that X is abelian. Since G/X is a p-group, it follows that X is self-centralizing.

Let λ be a linear constituent of χ_X, and let $S = \{s \in G \mid \lambda^s = \lambda^\sigma$ for some $\sigma \in \mathrm{Gal}(F(\lambda)/F)\}$. Note that S is a subgroup of G and that S contains the stabilizer, T, of λ in G. By 9.11 of [6], it follows that there exists a unique $\psi \in \mathrm{Irr}(T)$ such that $\psi^G = \chi$ and ψ_X has λ as a constituent. Let $\eta = \psi^S$.

We claim that $F(\eta) = F$. To prove this, let $\sigma \in \mathrm{Gal}(F(\eta, \lambda)/F)$. We show that $\eta^\sigma = \eta$. Now $\chi^\sigma = \chi$ and λ^σ is a constituent of χ_X so that $\lambda^\sigma = \lambda^y$ for some $y \in G$. By definition, $y \in S$. Since λ and $\lambda^\sigma = \lambda^y$ have the same stabilizer, T, in G, it follows that $T^y = T$ and $(\psi^\sigma)^{y^{-1}} \in \mathrm{Irr}(T)$. Now

$$((\psi^\sigma)^{y^{-1}})^G = (\psi^\sigma)^G = \chi^\sigma = \chi$$

and $\lambda = (\lambda^\sigma)^{y^{-1}}$ is a constituent of $((\psi^\sigma)^{y^{-1}})_X$. The uniqueness of ψ forces $\psi = (\psi^\sigma)^{y^{-1}}$ and since $y \in S$ we have

$$\eta = \psi^S = ((\psi^\sigma)^{y^{-1}})^S = (\psi^\sigma)^S = \eta^\sigma$$

as desired. Since $|F(\eta) : F| = 1$ and $[\chi, \eta^G] = 1$, we conclude that S is not a proper subgroup.

Thus all of the linear constituents of χ_X are Galois conjugate over F. In particular, they all have the same kernel. Since χ is faithful, λ is faithful and X is cyclic. Thus (G, X, χ) is an F-triple and as G/X is a p-group, the proof is complete.

5. The Main Theorem

Proof of Theorem 1. Assume, by way of contradiction, that some irreducible character of G has Schur index divisible by p. By Theorem 2, there is an F-triple (H, X, θ) such that $p \mid m_F(\theta)$, H/X is a p-group, and H is a section of G. Set $r = |X|$, $F_1 = F(\theta)$ and $E = F[\sqrt[r]{1}]$ so that the associated field extension is E/F_1. Since X is a cyclic section of G, we have $r \mid n$ and $\mathrm{Gal}(E/F_1)$ is a section of $\mathrm{Gal}(F[\sqrt[n]{1}]/F)$. By (2.1), the p-group H/X is isomorphic to $\mathrm{Gal}(E/F_1)$ and thus is cyclic.

Choose a supplement, Y, for X in H such that Y is a cyclic p-group and set $p^b = |Y|$ and $p^a = |Y \cap X|$. By (2.2), we may assume that $a > 0$. Since θ is faithful and $Y \cap X$ is central in H, we conclude that F_1 contains a primitive p^ath root of unity. We argue that in fact E contains a primitive p^bth root of unity.

Set $K = F_1[\sqrt[p^b]{1}]$. Since $p^b \mid n$, we see that E and K are intermediate fields lying between F_1 and $F[\sqrt[n]{1}]$. Moreover,

$$| E : F_1 | = | H/X | = | Y/Y \cap X | = p^{b-a},$$

and since F_1 contains a primitive p^ath root of unity for $a > 0$, we have

$$| K : F_1 | = p^e \leqslant p^{b-a}.$$

Now $\mathrm{Gal}(F[\sqrt[n]{1}]/F_1)$ has a cyclic Sylow p-subgroup. It follows that the fields between F_1 and $F[\sqrt[n]{1}]$ whose degree over F_1 is a power of p are linearly ordered, and we conclude that $K \subseteq E$ as asserted.

Let δ be a primitive p^bth root of unity in E. We claim that $N_{E/F_1}(\delta)$ is a primitive p^ath root of unity. Let $\mathrm{Gal}(E/F_1) = \langle\sigma\rangle$ so that $\delta^\sigma = \delta^k$ for some integer k. Then

$$N_{E/F_1}(\delta) = \prod_{i=0}^{p^{b-a}-1} \delta^{k^i} = \delta^q,$$

where $q = (k^{p^{b-a}} - 1)/(k - 1)$. Thus it suffices to show that p^{b-a} is the exact power of p dividing q. Let ϵ be a primitive p^cth root of unity in F_1 for some $c \geqslant a$. Because we are assuming that $\sqrt{-1} \in F$ when $p = 2$, we may assume $p^c > 2$. Since $\epsilon = \epsilon^\sigma = \epsilon^k$, we have $p^c \mid (k - 1)$ and it follows easily that p^{b-a} is the exact power of p dividing q.

We have now proved that the image of N_{E/F_1} contains a primitive p^ath root of unity and therefore $m_F(\theta) = 1$ by (2.2). This contradiction completes the proof of Theorem 1.

6. Concluding Remarks

If we drop the assumption that $\sqrt{-1} \in F$ when $| P |$ is even, then the theorem becomes false as the example $G = Q_8$ shows. Of course, we can conclude that $4 \nmid m_F(\chi)$ for $\chi \in \mathrm{Irr}(G)$ by applying Theorem 1 to the field $F[\sqrt{-1}]$.

If the exponent of G is not divisible by 4, then we can drop the assumption that $\sqrt{-1} \in F$. To see this, note that following the proof of Theorem 1, we need to consider an F-triple (H, X, θ), where $H = XY$, and Y is a cyclic 2-group. Thus $| Y | \leqslant 2$ and $X \cap Y = 1$. In this situation $m_F(\theta) = 1$ and the proof is complete.

Suppose we weaken the hypothesis that $\sqrt{-1} \in F$ when $| P |$ is even to the assumption that -1 is a sum of two squares. Is this sufficient to prove

$2 \nmid m_F(\chi)$ for $\chi \in \mathrm{Irr}(G)$? It is not hard to show that in this situation it suffices to consider F-triples (H, X, θ), where $H = XY$, for a cyclic 2-group Y with $|X \cap Y| = 2$ and $|X| = 2q$ for some odd prime q. The authors have been unable to settle this case.

Finally, we observe that the methods of this paper give an immediate proof of the following result of Fein–Yamada [5]: For any field F of characteristic zero and any irreducible character θ of G, $m_F(\theta)$ divides the exponent of G and $m_F(\theta)^2$ divides the order of G.

Proceeding by induction, we may assume by Theorem 2 that (G, X, θ) is an F-triple for some cyclic subgroup X. Then by the remark stated in Section 2, $m_F(\theta) \mid |X|$. Since we also know that $m_F(\theta) \mid \theta(1)$ and $\theta(1) = |G/X|$, the result follows.

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